A PROPERTY OF THE COEFFICIENT OF VARIATION AND ITS USE IN TRANSFORMATION OF DATA

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Abstract

The main use of the coefficient of variation to date has been to assess the variability in material in per cent of the mean value of the experiment or survey. In this paper we prove the following theorem concerning the coefficient of variation: "Let x be a random variable $N(\mu,\sigma^2)$, $\mu>0$, $\sigma>0$ and let x^k be the transformation of x for $k=1,2,3,\cdots$. Then, the coefficient of variation of x^k increases as k increases provided $3\mu^4>\sigma^4$." From empirical results (not included in the paper) and from some theoretical and specific points, it is conjectured that a minimum value of the coefficient of variation is attained when x^k (k = any real value and x = random variable) is normally distributed. Thus, this statistic may be used as a criterion for selecting a transformation to satisfy the normality condition in the analysis of variance.

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Introduction

In order to have valid results in the analysis of variance certain assumptions are to be satisfied. Eisenhart [1] pointed out these assumptions clearly. The difficulties arising from the failure of these assumptions were discussed by Cochran [2]. In order to fulfill the assumptions some kind of transformation is recommended according to the distribution underlying the observations. Bartlett [3] discussed some of the transformations. Generally the transformation to be used is known from the relation between mean and variance. Sometimes it is difficult to pin point a transformation according to the above rules as there will be some other difficulties in observing the data.

Results

Notation: Let x be a random variable distributed normally with the mean μ and variance σ^2 $(N(\mu,\sigma^2))$ where $\mu > 0$ and $\sigma > 0$. Let k be an integer greater than or equal to 1 (i.e., k=1,2,3,$\cdots$). Let α_k be the k^{th} moment of the random variable x

$$\left(\text{i.e.,}\quad \alpha_k = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^k e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \right)$$

for k=1,2,3,$\cdots$. α_0 is defined as equal to 1 and α_{-k} is defined as equal to zero

Lemma: If x is a random variable distributed $N(\mu,\sigma^2)(\sigma > 0)$ then $\alpha_k = \mu\alpha_{k-1} + (k-1)\sigma^2\alpha_{k-2}$

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Proof:

$$\alpha_k = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x^k e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $x-\mu=y$; then $x=y+\mu$ and $dx=dy$. Hence

$$\alpha_k = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (y+\mu)^k e^{-\frac{1}{2\sigma^2}y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (y+\mu)^{k-1} y\, e^{-\frac{1}{2\sigma^2}y^2} dy + \frac{\mu}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (y+\mu)^{k-1} e^{-\frac{1}{2\sigma^2}y^2} dy$$

Integrating by parts the first part of the right hand side we get

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (y+\mu)^{k-1} y\, e^{-\frac{1}{2\sigma^2}y^2} dy = \frac{-\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (y+\mu)^{k-1} d\, e^{-\frac{1}{2\sigma^2}y^2}$$

$$= \frac{-\sigma}{\sqrt{2\pi}} \left[(y+\mu)^{k-1} e^{-\frac{1}{2\sigma^2}y^2}\right]_{-\infty}^{\infty} + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (k-1)(y+\mu)^{k-2} e^{-\frac{1}{2\sigma^2}y^2} dy$$

$$= \sigma^2(k-1) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (y+\mu)^{k-2} e^{-\frac{1}{2\sigma^2}y^2} dy$$

$$= \sigma^2(k-1)\alpha_{k-2} \,.$$

The second part is $\mu\alpha_{k-1}$. Hence $\alpha_k=\mu\alpha_{k-1}+(k-1)\sigma^2\alpha_{k-2}$.

If x is a random variable then $f(x)$ is also a random variable and its expectation is

$$Ef(x) = \int f(x)dF \quad \text{where F is the distribution function} \tag{1}$$

of the random variable x.

$$E[f(x)]^2 = \int [f(x)]^2\, dF \tag{2}$$

variance of $f(x)$ is

$$E[f(x)]^2-[Ef(x)]^2 \tag{3}$$

The standard deviation is the square root of (3).

The coefficient of variation is defined as $s/\bar{x}$ where s is the standard deviation and $\bar{x}$ is the mean of the observed sample. As s^2 and $\bar{x}$ are unbiased estimators of σ^2 and μ the coefficient of variation can be written as σ/μ (i.e., standard deviation divided by mean for any $f(x)$).

Theorem: Let x be a randomvariable $N(\mu,\sigma^2)$, $\mu>0$, $\sigma>0$ and let x^k be the transformation of x for k=1,2,3,•••. Then, the coefficient of variation of x^k increases as k increases provided $3\mu^4>\sigma^4$.

Proof: We have to prove

$$\frac{[\alpha_{2k+2}-\alpha_{k+1}^2]^{1/2}}{\alpha_{k+1}} > \frac{[\alpha_{2k}-\alpha_k^2]^{1/2}}{\alpha_k} \qquad \text{for } k=1,2,3,\cdots$$

Because all these moments are positive since $\mu>0$ we look at

$$\frac{\alpha_{2k+2}-\alpha_{k+1}^2}{\alpha_{k+1}^2} > \frac{\alpha_{2k}-\alpha_k^2}{\alpha_k^2}$$

Rewriting the above,

$$\frac{\alpha_{2k+2}}{\alpha_{k+1}^2}-1>\frac{\alpha_{2k}}{\alpha_k^2}-1$$

and

$$\frac{\alpha_{2k+2}}{\alpha_{k+1}^2}>\frac{\alpha_{2k}}{\alpha_k^2}$$

If we can prove that $\alpha_{2k+2}\cdot\alpha_k^2-\alpha_{2k}\cdot\alpha_{k+1}^2>0$, this is sufficient to prove the theorem.

From the Lemma we have

$$\alpha_{2k+2} = \mu\alpha_{2k+1}+(2k+1)\sigma^2\alpha_{2k}$$

$$= \mu\left\{\mu\alpha_{2k}+(2k)\sigma^2\alpha_{2k-1}\right\}+(2k+1)\sigma^2\alpha_{2k}$$

$$= \left\{\mu^2+(2k+1)\sigma^2\right\}\alpha_{2k}+(2k)\sigma^2\mu\alpha_{2k-1}$$

$$\alpha_{k+1} = \mu\alpha_k+k\sigma^2\alpha_{k-1}$$

$$\alpha_{k+1}^2 = \mu^2\alpha_k^2+2k\mu\sigma^2\alpha_k\alpha_{k-1}+k^2\sigma^4\alpha_{k-1}^2$$

Hence

$$\alpha_{2k+2}\cdot\alpha_k^2 = \left[\left\{\mu^2+(2k+1)\sigma^2\right\}\alpha_{2k}+(2k)\mu\sigma^2\alpha_{2k-1}\right]\cdot\alpha_k^2$$

$$\alpha_{2k}\cdot\alpha_{k+1}^2 = \alpha_{2k}\cdot\left[\mu^2\alpha_k^2+2k\mu\sigma^2\alpha_k\cdot\alpha_{k-1}+k^2\sigma^4\alpha_{k-1}^2\right]$$

Hence

$$\alpha_{2k+2}\cdot\alpha_k^2-\alpha_{2k}\cdot\alpha_{k+1}^2 = \left\{\mu^2+(2k+1)\sigma^2\right\}\alpha_{2k}\cdot\alpha_k^2+(2k)\mu\sigma^2\alpha_{2k-1}\cdot\alpha_k^2$$

$$-\left[\mu^2\alpha_{2k}\cdot\alpha_k^2+2k\mu\sigma^2\alpha_{2k}\cdot\alpha_k\cdot\alpha_{k-1}+k^2\sigma^4\alpha_{2k}\cdot\alpha_{k-1}^2\right]$$

$$= (2k+1)\sigma^2\alpha_{2k}\alpha_k^2+(2k)\mu\sigma^2\alpha_{2k-1}\cdot\alpha_k^2-2k\mu\sigma^2\alpha_{2k}\cdot\alpha_k\cdot\alpha_{k-1}-k^2\sigma^4\alpha_{2k}\cdot\alpha_{k-1}^2$$

$$= (2k+1)\sigma^2\alpha_{2k}\cdot\alpha_k\left[\mu\alpha_{k-1}+(k-1)\sigma^2\alpha_{k-2}\right]+(2k)\mu\sigma^2\alpha_{2k-1}\cdot\alpha_k^2$$

$$-(2k)\mu\sigma^2\alpha_{2k}\cdot\alpha_k\cdot\alpha_{k-1}-k^2\sigma^4\alpha_{2k}\cdot\alpha_{k-1}^2$$

$$= \sigma^2\mu\alpha_{2k}\cdot\alpha_k\cdot\alpha_{k-1}+(2k+1)(k-1)\sigma^4\alpha_{2k}\cdot\alpha_k\cdot\alpha_{k-2}+(2k)\mu\sigma^2\alpha_{2k-1}\cdot\alpha_k^2$$

$$-k^2\sigma^4\alpha_{2k}\cdot\alpha_{k-1}^2 \tag{4}$$

Now let us divide the problem into two parts (i) k is even, (ii) k is odd.
Case (i) k is even: i.e., $k=2,4,6,\cdots$

$$\alpha_{k-1}^2 \leq \beta_{k-1}^2 \leq \beta_k\cdot\beta_{k-2} = \alpha_k\cdot\alpha_{k-2}$$

where β_k equal absolute k^{th} moment.

Hence substituting $\alpha_k \cdot \alpha_{k-2}$ for α_{k-1}^2 in (4), we get

$$\alpha_{2k+2} \cdot \alpha_k^2 - \alpha_{2k} \cdot \alpha_{k+1}^2 \geq \sigma^2 \mu \alpha_{2k} \cdot \alpha_k \cdot \alpha_{k-1}$$

$$+ (k^2-k-1)\sigma^4 \cdot \alpha_{2k} \cdot \alpha_k \cdot \alpha_{k-2} + (2k)\mu\sigma^2 \alpha_{2k-1} \cdot \alpha_k^2 > 0$$

Note that when k is even, the condition $3\mu^4 > \sigma^4$ is not necessary.

For example, let us consider the cases when k=2 and k=4. When k=2 we have to show that the coefficient of variation of x^3 is greater than the coefficient of variation x^2.

$$Ex^2 = \mu^2 + \sigma^2$$

$$Ex^3 = \mu^3 + 3\mu\sigma^2$$

$$Ex^4 = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$$

$$Ex^6 = 15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6$$

We have to show that

$$\frac{Ex^6}{(Ex^3)^2} > \frac{Ex^4}{(Ex^2)^2} \tag{5}$$

That is,

$$\frac{(15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6)}{(\mu^3 + 3\mu\sigma^2)^2} > \frac{(3\sigma^4 + 6\mu^2\sigma^2 + \mu^4)}{(\mu^2 + \sigma^2)^2}$$

or

$$(15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6)(\mu^4 + 2\mu^2\sigma^2 + \sigma^4) - (3\sigma^4 + 6\mu^2\sigma^2 + \mu^4)(\mu^6 + 6\mu^4\sigma^2 + 9\mu^2\sigma^4) > 0$$

Expanding we obtain,

$$120\mu^4\sigma^6 + 76\mu^6\sigma^4 + 17\mu^8\sigma^2 + \mu^{10} + 75\mu^2\sigma^8 + 15\sigma^{10} - 72\mu^4\sigma^6 - 48\mu^6\sigma^4 - 12\mu^8\sigma^2 - \mu^{10}$$

$$-27\mu^2\sigma^8 > 0$$

or,

$$15\sigma^{10}+48\sigma^{8}\mu^{2}+48\sigma^{6}\mu^{4}+28\sigma^{4}\mu^{6}+5\sigma^{2}\mu^{8} > 0$$

which is obviously true.

When k=4 we have to show that the coefficient of variation of x^5 is greater than coefficient of variation of x^4. That is, we have to show that

$$\frac{Ex^{10}}{(Ex^5)^2} > \frac{Ex^8}{(Ex^4)^2} \tag{6}$$

$$Ex^4 = 3\sigma^4+6\mu^2\sigma^2+\mu^4$$

$$Ex^5 = 15\mu\sigma^4+10\mu^3\sigma^2+\mu^5$$

$$Ex^8 = 105\sigma^8+420\mu^2\sigma^6+210\mu^4\sigma^4+28\mu^6\sigma^2+\mu^8$$

$$Ex^{10} = 945\sigma^{10}+4725\mu^2\sigma^8+3150\mu^4\sigma^6+630\mu^6\sigma^4+45\mu^8\sigma^2+\mu^{10}$$

Then (6) becomes

$$\frac{(945\sigma^{10}+4725\mu^2\sigma^8+3150\mu^4\sigma^6+630\mu^6\sigma^4+45\mu^8\sigma^2+\mu^{10})}{(15\mu\sigma^4+10\mu^3\sigma^2+\mu^5)^2}$$

$$> \frac{(105\sigma^8+420\mu^2\sigma^6+210\mu^4\sigma^4+28\mu^6\sigma^2+\mu^8)}{(3\sigma^4+6\mu^2\sigma^2+\mu^4)^2}$$

or,

$$(945\sigma^{10}+4725\mu^2\sigma^8+3150\mu^4\sigma^6+630\mu^6\sigma^4+45\mu^8\sigma^2+\mu^{10})$$

$$\cdot(9\sigma^8+36\mu^2\sigma^6+42\mu^4\sigma^4+12\mu^6\sigma^2+\mu^8)$$

$$>(105\sigma^8+420\mu^2\sigma^6+210\mu^4\sigma^4+28\mu^6\sigma^2+\mu^8)(225\mu^2\sigma^8+300\mu^4\sigma^6+130\mu^6\sigma^4$$

$$+20\mu^8\sigma^2+\mu^{10})$$

and,

$$8505\sigma^{18}+76545\mu^2\sigma^{16}+238140\mu^4\sigma^{14}+328860\mu^6\sigma^{12}+213030\mu^8\sigma^{10}+70614\mu^{10}\sigma^8$$

$$+12636\mu^{12}\sigma^6+1212\mu^{14}\sigma^4+57\mu^{16}\sigma^2+\mu^{18}-23625\mu^2\sigma^{16}-126000\mu^4\sigma^{14}$$

$$-186900\mu^6\sigma^{12}-126000\mu^8\sigma^{10}-44430\mu^{10}\sigma^8-8560\mu^{12}\sigma^6-900\mu^{14}\sigma^4$$

$$-48\mu^{16}\sigma^2-\mu^{18} > 0$$

Rewriting we obtain,

$$8505\sigma^{18}+52920\mu^2\sigma^{16}+112140\mu^4\sigma^{14}+141960\mu^6\sigma^{12}+87030\mu^8\sigma^{10}+26184\mu^{10}\sigma^8$$

$$+4076\mu^{12}\sigma^6+312\mu^{14}\sigma^4+9\mu^{16}\sigma^2 > 0$$

which is obviously true.

Case (ii): k is odd i.e., $k=1,3,5,\cdots$

All the moments can be expressed in terms of μ and σ. Thus, we may write

$$\alpha_{2k+2} = \sum_{r=0}^{k+1} \binom{2k+2}{2r}\mu^{2r}\cdot\sigma^{2k+2-2r} \cdot 1\cdot3\cdot5\cdots(2k+1-2r)$$

$$\alpha_{2k} = \sum_{r=0}^{k} \binom{2k}{2r}\mu^{2r}\cdot\sigma^{2k-2r} \cdot 1\cdot3\cdot5\cdots(2k-2r-1)$$

$$= \sum_{r=0}^{k+1} \binom{2k}{2r-2}\mu^{2r-2}\cdot\sigma^{2k+2-2r} \cdot 1\cdot3\cdot5\cdots(2k+1-2r)$$

$$\alpha_{k+1} = \sum_{j=0}^{\frac{k+1}{2}} \binom{k+1}{2j}\mu^{2j}\cdot\sigma^{k+1-2j} \cdot 1\cdot3\cdot5\cdots(k-2j)$$

$$\alpha_{k+1}^2 = \sum_{j=0}^{\frac{k+1}{2}} \sum_{h=0}^{\frac{k+1}{2}} \binom{k+1}{2j}\binom{k+1}{2\mu}\mu^{2j+2h}\cdot\sigma^{2k+2-2j-2h} \cdot 1\cdot3\cdot5\cdots(k-2j)$$

$$\times\ 1\cdot3\cdot5\cdots(k-2\mu)$$

$$\alpha_k = \sum_{j=1}^{\frac{k+1}{2}} \binom{k}{2j-1} \mu^{2j-1} \cdot \sigma^{k+1-2j} \cdot 1 \cdot 3 \cdot 5 \cdots (k-2j)$$

$$= \sum_{j=0}^{\frac{k+1}{2}} \binom{k}{2j-1} \mu^{2j-1} \cdot \sigma^{k+1-2j} \cdot 1 \cdot 3 \cdot 5 \cdots (k-2j)$$

$$\alpha_k^2 = \sum_{j=0}^{\frac{k+1}{2}} \sum_{h=0}^{\frac{k+1}{2}} \binom{k}{2j-1} \binom{k}{2\mu-1} \mu^{2j+2h-2} \cdot \sigma^{2k+2-2j-2h} \cdot 1 \cdot 3 \cdot 5 \cdots (k-2j)$$
$$\times\ 1 \cdot 3 \cdot 5 \cdots (k-2\mu)$$

Note: When r, j and h take their maximum limit the coefficient $1 \cdot 3 \cdot 5 \cdots (\cdots)$ is to be taken as 1 and not as -1 since the power of σ becomes zero the coefficient exists as 1 only. But when $0 \leq r < k+1$, $0 \leq j < \frac{k+1}{2}$, $0 \leq \mu < \frac{k+1}{2}$, the coefficient $1 \cdot 3 \cdot 5 \cdots (\cdots)$ is to be taken as usual.

$$\alpha_{2k+2} \cdot \alpha_k^2 - \alpha_{2k} \cdot \alpha_{k+1}^2$$

$$= \sum_{rjh}\sum\sum \binom{2k+2}{2r} \binom{k}{2j-1} \binom{k}{2\mu-1} \mu^{2j+2h+2r-2} \cdot \sigma^{4k+4-2r-2j-2h}$$
$$1 \cdot 3 \cdot 5 \cdots (2k+1-2r) \times 1 \cdot 3 \cdot 5 \cdots (k-2j) \times 1 \cdot 3 \cdot 5 \cdots (k-2\mu)$$

$$-\sum_{rjh}\sum\sum \binom{2k}{2r-2} \binom{k+1}{2j} \binom{k+1}{2\mu} \mu^{2j+2h+2r-2} \cdot \sigma^{4k+4-2j-2h-2r}$$
$$1 \cdot 3 \cdot 5 \cdots (2k+1-2r) \times 1 \cdot 3 \cdot 5 \cdots (k-2j) \times 1 \cdot 3 \cdot 5 \cdots (k-2\mu)$$

$$= \sum_{rjh}\sum\sum 1 \cdot 3 \cdot 5 \cdots (2k+1-2r) \times 1 \cdot 3 \cdot 5 \cdots (k-2j) \times 1 \cdot 3 \cdot 5 \cdots (k-2\mu) \mu^{2j+2h+2r-2}$$
$$\cdot \sigma^{4k+4-2r-2j-2h} \left[\binom{2k+2}{2r} \binom{k}{2j-1} \binom{k}{2\mu-1} - \binom{2k}{2r-2} \binom{k+1}{2j} \binom{k+1}{2\mu} \right] \qquad (7)$$

In $\alpha_{2k} \cdot \alpha_{k+1}^2$ there is a term $\sigma^{4k+2} 1 \cdot 3 \cdot 5 \cdots (k) \times 1 \cdot 3 \cdot 5 \cdots (k) \times 1 \cdot 3 \cdot 5 \cdots (2k-1)$ which is not in $\alpha_{2k+2} \cdot \alpha_k^2$. Hence consider the terms involving μ and σ in the above summation. The first thing that is required to be proved is that the terms involving μ and σ are positive in (7). After this a condition can be imposed which makes (7) definitely positive.

In order to prove that the terms involving μ and σ in (7) are positive it is sufficient to consider the coefficient

$$\binom{2k+2}{2r}\binom{k}{2j-1}\binom{k}{2\mu-1}-\binom{2k}{2r-2}\binom{k+1}{2j}\binom{k+1}{2\mu} \tag{8}$$

and show that it is positive. After some simplification (8) turns out to be

$$(2k+1)4j\mu-(k+1)r(2r-1) \tag{9}$$

Treating (9) as quadratic in r it is required to show that

$$2(k+1)r^2-(k+1)r-(2k+1)4j\mu \leq 0 \tag{10}$$

For (10) to be negative, the discriminant should be positive which is true and r should lie in the interval $[0,k+1]$ (i.e., $0 \leq r \leq k+1$). Because these are the limits of summation, r does lie between 0 and k+1. Hence negativeness of (10) implies positiveness of (9).

In (7) the terms $\mu^2\sigma^{4k}$ and μ^{4k+2} vanish. From (7) the term $\mu^4\sigma^{4k-2}$ has the coefficient $\frac{(7k^2+8k+3)}{6}1\cdot3\cdot5\cdots(2k-1)1\cdot3\cdot5\cdots k \times 1\cdot3\cdot5\cdots k$. When k=1 terms containing $\mu^4\sigma^2$ and σ^6 only remain. Hence it is proper to consider these two terms only in order to have a condition to make (7) strictly positive. Considering the terms $\frac{(7k^2+8k+3)}{6}\ 1\cdot3\cdot5\cdots(2k-1)1\cdot3\cdot5\cdots k \times 1\cdot3\cdot5\cdots k\mu^4\sigma^{4k-2}$ and $1\cdot3\cdot5\cdots(2k-1)1\cdot3\cdot5\cdots(k) \times 1\cdot3\cdot5\cdots k\sigma^{4k+2}$ we get the condition that $\frac{(7k^2+8k+3)}{6}\mu^4>\sigma^4$. When k=1 this condition turns out to be $3\mu^4>\sigma^4$ which is the condition of the theorem. Rewriting the inequality as $\mu^4>\frac{6}{7k^2+8k+3}\sigma^4$, we note that for sufficiently large k the inequality becomes essentially $\mu>0$.

To illustrate the above, two examples are given for k=1 and k=3.

Case (i) k=1

$$Ex = \mu \quad ; \quad Ex^2 = \mu^2+\sigma^2$$

$$; \quad Ex^4 = 3\sigma^4+6\mu^2\sigma^2+\mu^4$$

With the condition $3\mu^4>\sigma^4$ we will show that

$$\frac{Ex^4}{[Ex^2]^2} > \frac{Ex^2}{[Ex]^2} \tag{11}$$

Substituting the respective values in (11) we get

$$\frac{3\sigma^4+6\mu^2\sigma^2+\mu^4}{(\mu^2+\sigma^2)^2} > \frac{(\mu^2+\sigma^2)}{\mu^2}$$

$$3\mu^2\sigma^4+6\mu^4\sigma^2+\mu^6-\mu^6-3\mu^4\sigma^2-3\mu^2\sigma^4-\sigma^6 > 0$$

$$3\mu^4\sigma^2-\sigma^6 > 0$$

$$\sigma^2(3\mu^4-\sigma^4) > 0$$

which is true because of the condition $3\mu^4 > \sigma^4$.

Case (ii) When k=3

$$Ex^3 = \mu^3+3\mu\sigma^2 \quad , \quad Ex^6 = 15\sigma^6+45\mu^2\sigma^4+15\mu^4\sigma^2+\mu^6$$

$$Ex^4 = 3\sigma^4+6\mu^2\sigma^2+\mu^4 \quad , \quad Ex^8 = 105\sigma^8+420\mu^2\sigma^6+210\mu^4\sigma^4+28\mu^6\sigma^2+\mu^8$$

$$\frac{Ex^8}{[Ex^4]^2} > \frac{Ex^6}{[Ex^3]^2} \tag{12}$$

Substituting the values in (12) we get

$$(105\sigma^8+420\mu^2\sigma^6+210\mu^4\sigma^4+28\mu^6\sigma^2+\mu^8)(\mu^6+6\mu^4\sigma^2+9\mu^2\sigma^4)$$

$$-(15\sigma^6+45\mu^2\sigma^4+15\mu^4\sigma^2+\mu^6)(9\sigma^8+36\mu^2\sigma^6+42\mu^4\sigma^4+12\mu^6\sigma^2+\mu^8) > 0$$

$$=2025\mu^4\sigma^{10}+1896\mu^6\sigma^8+711\mu^8\sigma^6+120\mu^{10}\sigma^4+7\mu^{12}\sigma^2-135\sigma^{14} > 0$$

which can easily be seen true with the condition that $3\mu^4 > \sigma^4$.

Note: As mentioned above, there is a term containing only a power of σ which becomes negative and the rest of the terms in (7) are positive. Also, the terms involving μ^2 and a power of σ and terms involving only a power of μ vanish. This can be observed in the above two examples.

Discussion

It is shown for positive integers of k that the coefficient of variation increases as k increases. Empirical results indicate that the coefficient of variation for x^k, for all real k, is a minimum for that power of x which is distributed $N(\mu,\sigma^2)$ for $\mu > 0$, $\sigma > 0$, and $3\mu^4 > \sigma^4$. Hence from the above results the following may be conjectured.

i) For all real k and for x distributed $N(\mu,\sigma^2)$, for $\mu > 0$, $\sigma > 0$, and $3\mu^4 > \sigma^4$, the coefficient of variation increases with k as k departs from unity in either direction.

ii) The appropriate transformation of the data to satisfy the normality condition in the analysis of variance is that power of x for which the coefficient of variation is a minimum.

The coefficient of variation may be used as a criterion to determine the appropriate transformation in the analysis of variance. The analysis of variance is performed on a number of powers of x around the power of x suspected to be normally distributed. The coefficients of variation are computed from each of these analyses and a quadratic regression of the value of the coefficient of variation on the power of x is fitted to the data. The minimum value of the quadratic function, i.e., $\hat{k} = -b/2c$ where b = linear regression coefficient and c = the quadratic regression coefficient, is taken as the power of x upon which the data are to be analyzed. It should be sufficient to take $\hat{k}$ equal to the nearest half integer.

Summary

The main use of the coefficient of variation to date has been to assess the variability in material in per cent of the mean value of the experiment or survey. In this paper we prove the following theorem concerning the coefficient of variation:

"Let x be a random variable $N(\mu,\sigma^2)$, $\mu > 0$, $\sigma > 0$ and let x^k be the transformation of x for $k=1,2,3,\cdots$. Then, the coefficient of variation of x^k increases as k increases provided $3\mu^4 > \sigma^4$." From empirical results (not included in the paper) and from some theoretical and specific points, it is conjectured that a minimum value of the coefficient of variation is attained when x^k (k = any real value and x = random variable) is normally distributed. Thus, this statistic may be used as a criterion for selecting a transformation to satisfy the normality condition in the analysis of variance.

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